

GEODESIC CORRESPONDENCES BETWEEN SURFACES OF REVOLUTION

BY

JAMES COLLIER STEWART

B.A., University of Mississippi, 1934

M.A., University of Mississippi, 1936

M.S., Louisiana State University, 1940

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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MATHEMATICS IN THE GRADUATE SCHOOL
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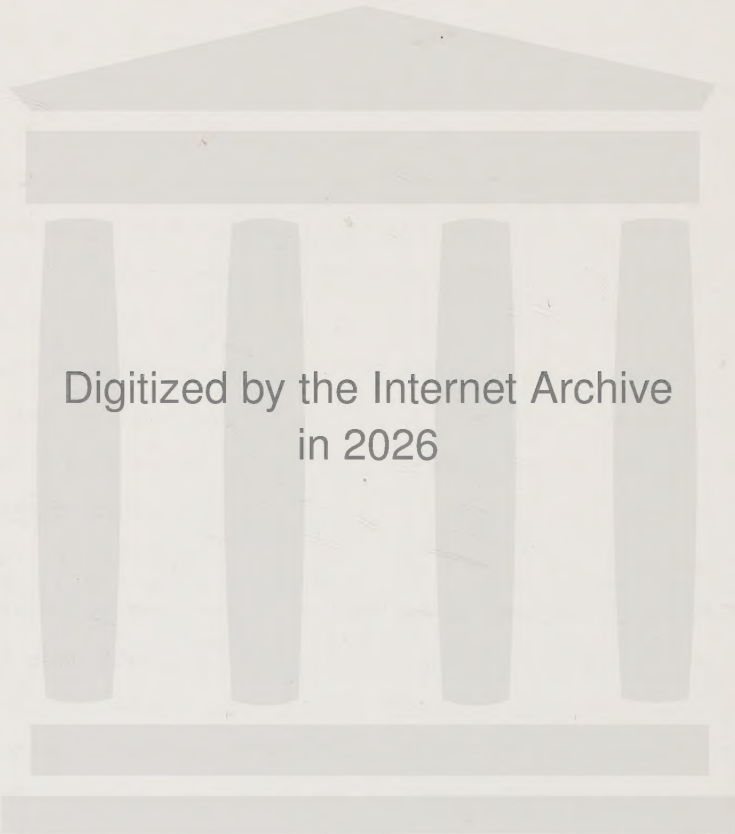
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A geodesic correspondence with, or mapping on, one surface of another is a one-to-one point-correspondence such that the geodesics of the first surface correspond to the geodesics of the second. The research of Eisenhart, Veblen, and others into the projective geometry of non-Riemannian spaces reduced the problem of determining the geodesic correspondences to the problem of solving a system of linear partial differential equations. This system is discussed on pages 100-102 of Eisenhart's Non-Riemannian Geometry. Using the integrability conditions of the system and other non-differential conditions, Levy* has shown that, on a surface not of constant Gaussian curvature, two vectors, K^1 and L^1 , can be defined each of which must correspond under geodesic mappings to a vector of the same kind on a second surface. The congruence of curves determined by the K-vector consists of the orthogonal trajectories of the curves along each of which the Gaussian curvature, K , is constant. (See Cartan's Leçons sur la Théorie des Espaces à Connexion Projective, pages 251-253.)

If the L-vector is linearly dependent on the K-vector, then the curves $K = \text{constant}$ are geodesic parallels. The condition is both necessary and sufficient. The system of geodesics orthogonal to the curves $K = \text{constant}$ forms an invariant congruence under geodesic mappings.

The surfaces on which the two vectors are linearly dependent are called H-surfaces. When the curves $K = \text{constant}$ and the geodesics orthogonal to these curves are

*See Bibliography.

chosen as the coordinate curves, $u = \text{constant}$ and $v = \text{constant}$, respectively, on an H-surface S, then the linear element of S can be put in the form

$$ds^2 = du^2 + H^2(V + U)^2 dv^2,$$

where H and U are functions of u alone, V is a function of v alone, and

$$U = c \int H^{-2} du + c_1,$$

c and c_1 being constants. Every surface whose linear element can be put in this form is in the set of H-surfaces. This set contains all the surfaces isometric to surfaces of revolution, including the surfaces of constant curvature.

If $\bar{S}$ is an H-surface, its linear element can be written

$$d\bar{s}^2 = dx^2 + \bar{H}^2(X + Y)^2 dy^2,$$

where $\bar{H}$ and X are functions of x alone, Y is a function of y alone, and

$$X = \bar{c} \int \bar{H}^{-2} dx + \bar{c}_1.$$

When coordinates on S and $\bar{S}$ are chosen as above, the curves $v = \text{constant}$ must correspond to the curves $y = \text{constant}$; the equations of the geodesic correspondences must be of the form

$$\begin{cases} x = p(u, v) \\ y = q(v) \end{cases}.$$

The differential equations discussed on page 1, which determine the functions p and q, become:

$$\begin{aligned} \frac{p_{uu}}{p_u} &= 2p_u \left[\frac{\bar{H}'}{\bar{H}} + \frac{X'}{Y + X} \right] - 2 \left[\frac{H'}{H} + \frac{U'}{V + U} \right] \\ \frac{p_{uv}}{p_u} &= 2p_v \left[\frac{\bar{H}'}{\bar{H}} + \frac{X'}{Y + X} \right] + \frac{1}{2} \left[\frac{Y'q'}{Y + X} + \frac{q''}{q'} - \frac{V'}{V + U} \right] \end{aligned}$$

$$\begin{aligned}
p_{vv} = & 2p_v^2 \left[\frac{\bar{H}'}{\bar{H}} + \frac{X'}{Y+X} \right] + p_v \left[\frac{Y'q'}{Y+X} + \frac{q''}{q'} \right] \\
& + q'^2 \left[\bar{H}'\bar{H}(Y+X)^2 + \bar{c}(Y+X) \right] \\
& - p_u \left[H'H(V+U)^2 + c(V+U) \right] .
\end{aligned}$$

The properties of the geodesic correspondences between H-surfaces can be found by the use of these differential equations and of other equations obtained from these by differentiation. From the integrability condition

$$\frac{Y'q'X'p_u}{(Y+X)^2} = \frac{V'U'}{(V+U)^2} ,$$

it follows that a surface isometric with a surface of revolution can be in geodesic correspondence only with another such surface. From the condition

$$\bar{K}'\bar{H}^2(Y+X)^2 q'^2 p_u = K'H^2(V+U)^2 ,$$

it follows that a surface of constant curvature can be in geodesic correspondence only with another surface of constant curvature.

We may here note that we use "surface" in the sense of a two-dimensional Riemannian space, and, therefore, by "surface of revolution" we mean a space possessing the linear element

$$ds^2 = du^2 + H^2(u)dv^2 .$$

An H-surface that is not a surface of revolution can be in geodesic correspondence only with another H-surface, and the two surfaces must allow linear elements whose ratio is a constant; that is, they must be similar surfaces. The correspondences must be such that the ratio of corresponding distances is constant.

If S is a surface of revolution, referred to the above

linear element, the curves $K = \text{constant}$ (the v-curves) are the parallels, and their orthogonal trajectories (the u-curves) are the meridians. There is a subset of surfaces of revolution which allow geodesic correspondences with S of the type that preserve parallels as well as meridians. The linear element of any surface $\bar{S}$ of the subset can be written

$$d\bar{s}^2 = \frac{a^2 d\bar{u}^2}{[1 + bH^2(\bar{u})]^2} + \frac{a^2 H^2(\bar{u}) d\bar{v}^2}{1 + bH^2(\bar{u})},$$

where a and b are arbitrary constants. The geodesic correspondences between S and $\bar{S}$ have the equations:

$$\begin{cases} \bar{u} = \int \frac{a_1 du}{1 + b_1 H^2(u)} + n_1 \\ y = mv + n, \end{cases}$$

with the restriction that

$$H^{-2}(\bar{u}) = m^2 a_1^{-2} [H^{-2}(u) + b_1].$$

At most three of the constants a_1 , b_1 , m , n , and n_1 can be arbitrary. The writer was unable to discover, however, any case in which more than two parameters occur in the equations of the correspondences between two surfaces of the same subset. Two parameters occur for the surfaces such that $H(u) = u^c$ ($c \neq 1$). In any case, there is one parameter, n .

Surfaces not of constant K such that the vector L^1 is a zero-vector are called P-surfaces. They must admit the linear elements

$$ds^2 = \frac{dz^2}{(z^3 + Cz + D)^2} + \frac{dv^2}{z^3 + Cz + D}.$$

All P-surfaces such that $C \neq 0$ form a subset of surfaces of revolution between which there are geodesic correspondences such that parallels are preserved. The equations of the

correspondences between two such surfaces have one parameter.

The P-surfaces such that $C = 0$ are called P_0 -surfaces. They also form a subset of surfaces of revolution that allow geodesic correspondences such that parallels are preserved. The equations of the correspondences between two P_0 -surfaces have two parameters.

A surface of revolution is in geodesic correspondence only with another surface of its subset unless it is a surface of constant curvature or a Q-surface.

A Q-surface is a surface of revolution not of constant K which allows geodesic correspondences with other surfaces in such a way that parallels are not preserved. Consequently, a Q-surface can be in geodesic correspondence only with other Q-surfaces. The linear element of a Q-surface S may be written

$$ds^2 = \frac{z^4 dz^2}{(1 + Cz^2 + Ez^4)^2} + \frac{z^2 dv^2}{1 + Cz^2 + Ez^4}.$$

If $\bar{S}$ is any other Q-surface, with linear element

$$d\bar{s}^2 = \frac{\bar{z}^4 d\bar{z}^2}{(1 + \bar{C}\bar{z}^2 + \bar{E}\bar{z}^4)^2} + \frac{\bar{z}^2 d\bar{v}^2}{1 + \bar{C}\bar{z}^2 + \bar{E}\bar{z}^4},$$

the equations of the geodesic correspondences of S with $\bar{S}$ are

$$\begin{cases} \bar{z} = \pm |q'(v)|^{\frac{1}{2}} z \\ \bar{v} = q(v) \end{cases},$$

where $q(v)$ is a solution of the differential equation

$$\frac{q'''}{q'} - \frac{3q''^2}{2q'^2} + 2\bar{E}q'^2 = 2\bar{E}.$$

When $E = 0$, the Q-surface is called a Q_0 -surface. In the above coordinates, the geodesic correspondences between two Q_0 -surfaces are three-parameter linear fractional transformations.

Between two surfaces of revolution not of constant curvature, there cannot be geodesic correspondences of more than three parameters.

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VITA

James Collier Stewart was born in Memphis, Tennessee, on March 30, 1913. He attended grammar school and high school in Charleston, Mississippi, and entered the University of Mississippi in the fall of 1931. In 1934, he received there the Bachelor of Arts degree with special distinction, and, in 1936, the Master of Arts degree with a major in Latin.

The next two years he spent as a teacher in Charleston High School of Latin and algebra, and the next two as a part-time teacher in the Department of Mathematics of Louisiana State University, where he received, in 1940, the degree of Master of Science with a major in mathematics.

His work toward the degree of Doctor of Philosophy in mathematics was begun in the fall of that year at the University of Illinois. He taught in the Department of Mathematics there for two years while going to school. Then he became an instructor of mathematics at Louisiana State University, where he has taught for the last four years. His summers since 1942 have been spent in graduate study at the University of Illinois.

